MBA Business Foundations, Quantitative Methods: Session Five

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Today

R vs. Excel

- Steeper learning curve (whether it's worth it, depends on your objectives!)
- Fast
- Free
- Flexible
- Reproducible
- Extremely well supported (stackexchange, rbloggers, etc.)
- Big data friendly (eg. no bound on rows/columns)
- Visualization and graphics! (with ggplot2)
- Data science (with the Tidyverse)
- Machine learning (with Caret, randomForest, nnet, e1071)
- Bayesian modeling/Markov Chain Monte Carlo (with Bugs, Jags, and Stan)

... you learn as you go/as needed, shouldn't try to "learn R"!

Summary so far

- Intro to functions (ordinary, inverse, quadratic...)
- Exponents and logs
- Derivatives
- Optimization
- Statistics and probability (last class: measures of location and dispersion)

Probability

- Tyche: Goddess of chance (daughter of Zeus).
- The ancient Greeks believed that when no other cause can be attributed to random events such as floods, droughts, frosts, then Tyche is responsible.
- Probability is the study of such random events or, more generally, of randomness.

Probability

- But what is probability?
- For our purposes, the probability of an event can be interpreted as the long-run frequency of the occurrence of that event.
- Always???
- What is the probability of a nuclear war in the next year? Of US dollar collapse?
- When Galileo first observed Saturn through a telescope, he saw something like this.

- Are those rings around the planet? Handles? Or is it three planets next to each other? Can you assign a probability?

Probability vs. statistics

- What is the difference between probability and statistics?
- A probability question: A fair die will be tossed twice. What is the probability that it lands on six both times?
- A (descriptive) statistics question: A die was tossed twice and it landed on a five and a six. What is the mean die value?
- An (inferential) statistics question: A die was tossed twice and it landed on a five and a six. How confident are you that the die is fair?
- To answer this question, can you use descriptive statistics? Can you use probability? How? Is there a right answer? What is it?

Random variables

- In probability we know which values our variables X can take, and we know how probable those values are.
- Ex: if the variable represents the outcome of the toss of a fair die (i.e., which face landed up), what are the values? How likely are they?
- Variables are
  - Discrete, corresponding to natural or counting numbers.
  - Continuous, corresponding to real numbers.
- Classify the following:
  - Height or weight
  - Number of monthly lottery winners in California
  - Temperature tomorrow
  - A parent's number of children
  - The amount of money in your bank account

Knowing your sample space

- A certain couple has two children. At least one of them is a boy. What is the probability that both children are boys?
- Possibilities: BB, BG, GB, GG
- What can we rule out? GG
- What remains: BB, BG, GB
- Probability that both children are boys is 1/3.

Probability distributions

- A probability distribution is a function f of the random variable X: $f(X)$
- This function can only take values between 0 and 1.
- Also, this function is additive: for two independent events, the probability of their sum is the sum of their probabilities.
- Ex: Pr (die lands on 4) + Pr (die lands on 6) = Pr (die lands on 4 or die lands on 6).
- In the discrete case, it tells us how probable it is that the random variable X will take a specific value x. We often denote the function f with $\Pr$.
- Ex: For a fair die, $f(3) = \Pr(X = 3) = 1/6$.
- In the continuous case, it tells us how likely it is that our variable will be contained within an interval $[a, b]$.
- Ex: For a person's weight, $\Pr(a < x < b) = k$.
- But what about $\Pr(x)$ in the continuous case (???).

Variations on exercise problem 4

- A fair die is rolled once.
- What is the probability that it lands on a number greater than 4?

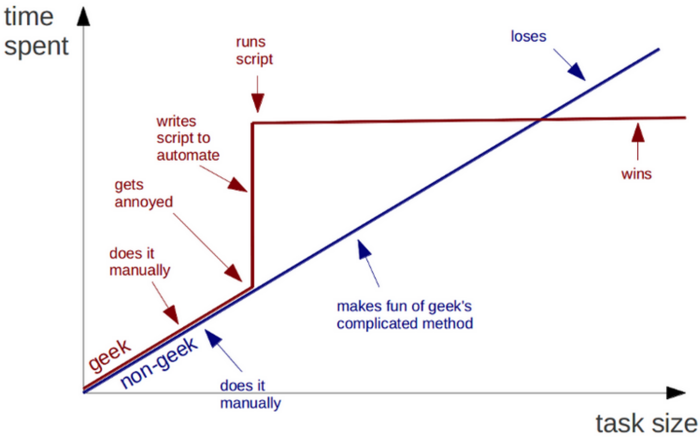
$$\Pr(X > 4) = \Pr(X = 5) + \Pr(X = 6) = 2/6 = 1/3$$

```
library(dice)
getEventProb(nrolls = 1,ndicePerRoll = 1,
nsidesPerDie = 6,eventList = list(5:6))
[1] 0.3333333
```

- A fair die is rolled 5 times. What is the probability of seeing exactly the pattern 6, 5, 4, 3, followed by a 2 or a 1?
- $\left(\frac{1}{6}\right)^5 + \left(\frac{1}{6}\right)^5 = 0.0002$

```
library(dice)
getEventProb(nrolls = 5,ndicePerRoll = 1,nsidesPerDie = 6,
eventList = list(6, 5, 4, 3, 1:2),orderMatters = TRUE)
[1] 0.0002572016
```

Doing things manually vs. automating them: geek vs. non-geek



Birthdays

- We have 30 students in this class. What is the probability that at least one pair of students share the same birthday?

- Algebraic solution:

  $\frac{30 \cdot 29}{2} = 435$ pairs of students

  $\frac{364}{365}$ probability that a single pair does not share a birthday

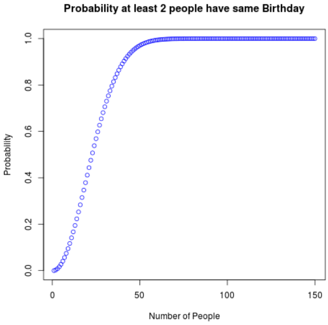
  $\left(\frac{364}{365}\right)^{435} = 0.30$ probability that no pair shares a birthday

  $1 - 0.30 = 0.70$ probability that at least one pair shares a birthday

Birthdays: R solution

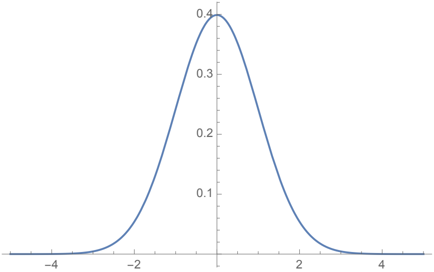
```
k = 30
p <- numeric(k)  # create numeric vector to store probabilities
for (i in 1:k)      {
            q <- 1 - (0:(i - 1))/365  # 1 - prob(no matches)
            p[i] <- 1 - prod(q)  }
prob <- p[k]
print(prob)
#BONUS:

plot(p, main="Probability at least 2 people have same Birthday",
xlab ="Number of People", ylab = "Probability", col="blue")
[1] 0.7063162
```



Normal distribution

- Most important probability distribution you will encounter (due, in part, to the central limit theorem).
- This distribution belongs to the exponential family of distributions, and it has two parameters, its average μ and standard deviation σ.
- Represented by the famous "bell curve": symmetric around its mean

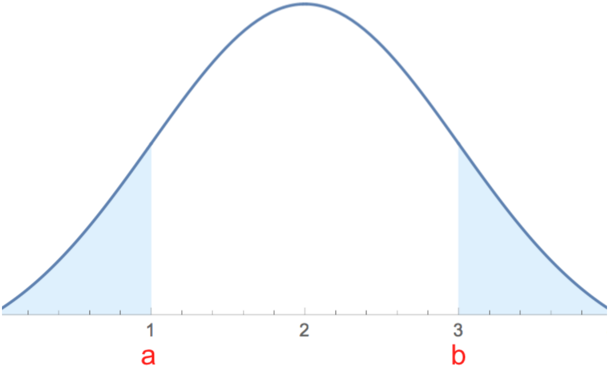


- Given by

$$f(x|\mu,\sigma^2) = (2\pi\sigma^2)^{-1/2} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

- Bonus: Why is π in there? Because $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$!

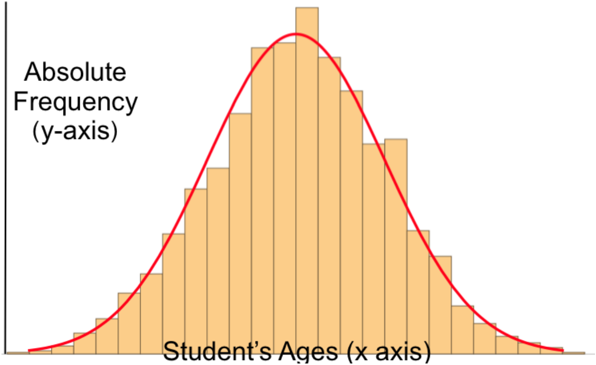
Normal distribution



The probability that our random variable is between a and b is given by the area under the curve between those two points:

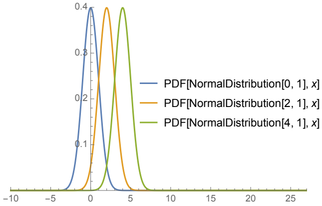
$$\Pr(a < x < b) = (2\pi\sigma^2)^{-1/2} \int_a^b e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

Normal distribution

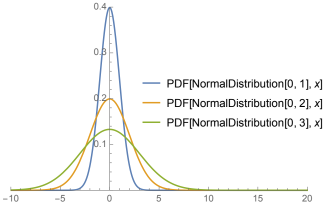


- Variables that have a normal distribution are ubiquitous in real life, provided we have enough data.
- Ex: Age of INSEAD students, height of INSEAD students.

Normal distribution



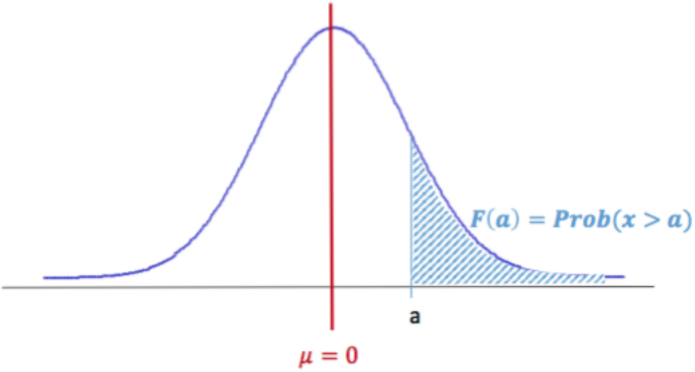
- As the mean changes, the location of the bell shifts
  To the left (for smaller means)
  To the right (for larger means)



- As the standard deviation changes, the bell becomes
  taller and thinner (for smaller standard deviations)
  shorter and thicker (for larger standard deviations)

Normal distribution

- So how to compute these areas under the curve (=probabilities)?
- The integral does not have a closed form.
- Rescale to a standard normal distribution and then use a table.
- Or, use computational approach, for example in R!



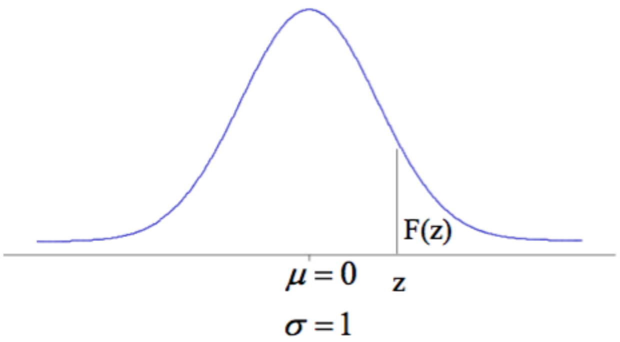
Transformations

- A standard normal distribution is a normal distribution with mean 0 and variance 1.
- The distribution function of a standard normal is given by

$$(2\pi)^{-1/2}e^{\frac{-x^2}{2}}$$

- To denote that a random variable X has a normal distribution we will use $X \sim \mathrm{N}(\mu, \sigma^2)$.
- if X follows a normal distribution with mean μ and standard deviation σ, $X \sim \mathrm{N}(\mu, \sigma^2)$, then $Z = \frac{x-\mu}{\sigma}$ follows a standard normal distribution, $Z \sim \mathrm{N}(0, 1)$.

Transformations

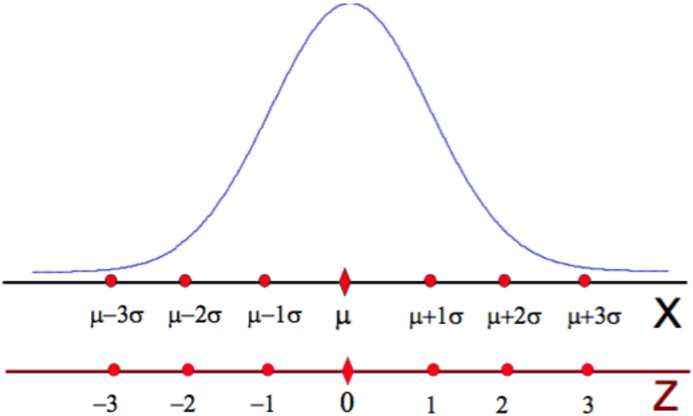


- Capital F denotes a cumulative distribution:

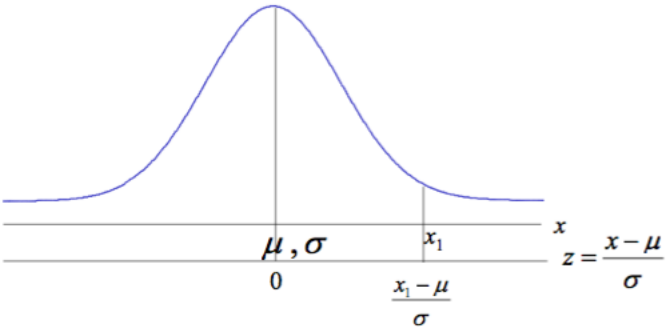
$$F(z) = Pr(Z \leq z) = \int_{-\infty}^{z} f(z)dz$$

- So $\Pr(a < x < b) = F(b) - F(a)$

Transformations



Transformations



Computing normal probabilities

- Ex: You want $\Pr(X > k)$ where $X \sim \mathrm{N}(\mu, \sigma)$.
- Step 1: Transform $X \to Z$

$$\begin{aligned}\Pr(X > k) &= \Pr\left(\frac{X-\mu}{\sigma} > \frac{k-\mu}{\sigma}\right)\\ &= \Pr\left(Z > \frac{k-\mu}{\sigma}\right)\end{aligned}$$

- Step 2: Look up the probability of $F(\frac{k-\mu}{\sigma})$ in R or a standard normal table.
- Step 3: Get the result:

$$\Pr(X > k) = F(\frac{k-\mu}{\sigma})$$

Examples

- Suppose that the test scores of a course exam at INSEAD are normally distributed with a mean of 72 and a standard deviation of 15.2. What is the probability that a randomly chosen student received above 84?

$$\Pr(X > 84) \to \Pr\left(\frac{X-\mu}{\sigma} > \frac{84-72}{15.2}\right) \to \Pr(Z > 0.789)$$

- pnorm(0.789, lower.tail=FALSE)
- 1-pnorm(0.789)
- OR, you can avoid the transformations altogether! pnorm(84, mean=72, sd=15.2, lower.tail=FALSE)
- Approximately 21%.
- We use lower.tail=FALSE in order to get the area from x to ∞.

The pnorm() function

- The pnorm function replaces the lookup table at the end of all statistics textbooks.
- pnorm returns the integral from $-\infty$ to k of the pdf of the normal distribution. That is, $F(k)$.
- If you do not set any further values, then k is a Z-score by default.
- However, you can specify the mean and variance as pnorm(2, mean = 5, sd = 3)

Practice

- The weekly salaries of the employees of a large corporation are assumed to be normally distributed with mean \$450 and standard deviation \$40.
- What is the probability that a randomly chosen employee earns more than \$500 per week?

Solution

- Find the Z score and look it up:

$$\Pr(X > 500) = \Pr\left(\frac{X - 450}{40} > \frac{500 - 450}{40}\right) = \Pr\left(Z > \frac{5}{4}\right)$$

- pnorm(500, mean=450, sd=40, lower.tail=FALSE)
- approximately 10%.

Tips

- Probabilities correspond to areas
- Probabilities sum to 1: $\Pr(Z < k) = 1 - \Pr(Z > k)$
- Symmetry: $\Pr(Z < -k) = \Pr(Z > k)$
- For intervals, use substraction: $\Pr(a < Z < b) = \Pr(Z > a) - \Pr(Z > b)$

Practice: but how much money will I make???

- The average post INSEAD MBA starting salary is $170k$ with a standard deviation of $30k$.
- You want to know the probability of earning between 150 to $200k$ upon graduation.
- So you want to calculate the probability that a randomly chosen post-MBA INSEAD student starts at $150 - 200k$.
- Find the answer algebraically, then confirm in R.

Solution

Algebraic:

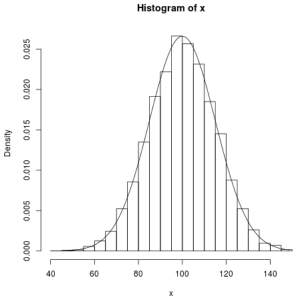
$$
\begin{aligned}
\Pr(150 < x < 200) &= \Pr(x > 150) - \Pr(x < 200) \\
&= \Pr\left(\frac{x-170}{30} > \frac{150-170}{30}\right) - \left(\frac{x-170}{30} > \frac{200-170}{30}\right) \\
&= \Pr\left(Z > \frac{150-170}{30}\right) - \left(Z > \frac{200-170}{30}\right) \\
&= \Pr\left(Z > -\frac{2}{3}\right) - \left(Z > 1\right) \\
&= .7454 - .1587 \\
&= .5867
\end{aligned}
$$

In R:

```
pnorm(150, mean=170, sd=30, lower.tail=FALSE) -
pnorm(200, mean=170, sd=30, lower.tail=FALSE)
```

The rnorm() function

```
x <- rnorm(10000, mean=100, sd=15)
hist(x, probability=TRUE)
xx <- seq(min(x), max(x), length=150)
lines(xx, dnorm(xx, mean=100, sd=15))
```



This generates 10000 random numbers from a specified normal distribution (first line), plots their histogram (second line), and graphs the distribution function of the same normal distribution (third and fourth lines).

Today

Basics	Functions Linear Inverse Two equations Quadratic
Exponents	Exponents Application: interest rates Exponential functions Logarithmic functions
Logarithms	Logarithmic functions Logarithmic and exponential equations Case: pricing Derivatives
Derivatives	Optimal decisions Case: production Statistics
Uncertainty	Probability & statistics Normal distribution

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