ALGEBRA AND CALCULUS FOR BUSINESS

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Exponential and Logarithmic Functions

The manager of the Plunket Company has noted that sales growth of their major product line has continued, but that the amount of the increase in sales of this line has been declining by approximately the same percentage for several periods. He is interested in estimating future sales levels. The Department of Health, Education, and Welfare is interested in how population growth in certain areas will affect particular programs and needs to know the impact of, say, a constant percentage growth rate on future population size. A firm or government agency is concerned with the depletion of resources (say, oil) available to it for use or sale. Another firm is concerned with the decay of the purchase rate of a product as one factor in the decision of whether to continue producing the product.

All of these situations and a host of other managerial decisions involve the notions of growth and decay. Where this growth or decay is in equal amounts per unit of, say, time for example, a linear function can be used to represent the phenomenon. For example, if sales start at 100 units today and increase by 5 units per month, then for any future month starting from the present,

$$\text{sales} = 100 + 5x, \qquad x \geq 0,$$

where x is in months and x equals 0 for the present month.

But quite often growth or decay is not adequately described by linear relationships. More complex curves are necessary, and these typically involve exponential or logarithmic functions.

We have already spent some time in Chapter 2 reviewing exponents. This chapter begins with a review of logarithmic operations before a more practical exploration of the growth and decay functions which they describe.

4-1 LOGARITHMS

In Chapter 2 we discussed the concept of an exponent and investigated several equations involving exponents. A simple example is

$$3^2 = 9.$$

This mathematical expression can be translated as: (1) 3 multiplied by itself gives (equals) 9; *or* (2) 3 squared is 9; *or* (3) 2 is the power to which 3 must be raised to yield 9.

Translation (3) is the one we shall focus on here. Mathematicians write this expression using the notation

$$2 = \log_3 9$$

and say "the logarithm of 9 to the base 3 is 2." Slightly expanded, and using statement (3), this expression becomes "the logarithm of 9 to the base 3 is the power (exponent) needed to raise 3 to 9." This power is, of course, 2. Hence a logarithm is an exponent.

Let's try another example.

$$4^3 = 64.$$

In logarithmic form the exponential expression becomes

$$3 = \log_4 64$$

and is phrased as "the logarithm of 64 to the base 4 is 3." This means that 3 is the power that raises 4 to 64. This is a direct translation of $3 = \log_4 64$. The correspondence is "3" in the phrase for the 3 in the mathematical expression, "is" for =, "the power which raises" for log, 4 for 4, and "to 64" in the phrase for 64 in the equation.

In more general terms:

$$x = \log_b a,$$

which means that x is the power needed to raise b to a. We write

$$b^x = a.$$

A logarithm is an exponent.

Symbol	*English Translation*
$x = \log_b a$	x is the exponent needed to raise b to the value a.

Logarithm as an Inverse Function

The two equations above the symbol translation suggest the dual or inverse relationship between equations written in logarithmic and exponential form. For example, suppose that we are given the logarithmic equation

$$x = \log_3 81$$

and asked to find x. You may already know the answer, but most of us find it simpler to understand what is going on by first converting this expression to its exponential equivalent (or inverse form). In inverse form we obtain

$$3^x = 81.$$

We suspect that the answer, $x = 4$, is more easily obtained using this form.

Exponential and logarithmic functions are connected by this inverse relationship. Given the exponential equation in general form

$$y \equiv f(x) = b^x,$$

we can write the inverse logarithmic equation

$$x \equiv f(y) = \log_b y.$$

Note that, as has now been illustrated, we can go either way.

: Write the following logarithmic equations in exponential form and solve for x.

	Exponential form	*Solution*
1. $x = \log_3 27$		$x =$
2. $x = \log_3 3$		$x =$
3. $x = \log_3 1$		$x =$
4. $x = \log_{0.1} 0.0001$		$x =$
5. $x = \log_b b^{2a}$		$x =$
6. $x = \log_4 128$		$x =$
7. $x = \log_{10} 0.1$		$x =$

Answers

1. $3^x = 27$	$x = 3$
2. $3^x = 3$	$x = 1$
3. $3^x = 1$	$x = 0$
4. $(0.1)^x = 0.0001$	$x = 4$
5. $b^x = b^{2a}$	$x = 2a$
6. $4^x = 128$	$x = 3.5$ since $128 = 4 \cdot 4 \cdot 4 \cdot 2 = 4^3 \cdot 4^{\frac{1}{2}} = 4^{3\frac{1}{2}}$
7. $10^x = 0.1$	$x = -1$ since $10^{-1} = \frac{1}{10} = 0.1$

Logarithmic Bases

The previous section indicates that any one of a number of bases can be used for logarithmic functions. Nevertheless, certain bases have found particular favor. The first of these is the base 10. This base is particularly useful in hand numerical calculation. The rules for calculations using logarithms are briefly reviewed in Section 4–4. The advent of electronic computers has made it much less necessary to make use of logarithmic hand calculations, however. (It is interesting to recall from Section 2–3 that the electronic computer uses a base other than 10, namely 2, for making calculations.)

The other major logarithmic base commonly found in practical applications is the mathematical constant $e = 2.71828\ldots$. This number arises often enough that logarithmic tables have been computed using the constant e as a base. Logarithms using the base e are called *natural logarithms*. Again, for simplification, the mathematician uses shorthand and writes ln for natural logarithm, rather than $\log_e$. In other words, $\ln \equiv \log_e$.

One example of where natural logarithms arise in a managerial setting is illustrated by the topic of compound interest which we studied in Section 2–2. At that time we noted that \$1.00 invested at an annual interest rate of r for t years accumulates to $(1 + r)^t$. We saw further that if interest is compounded semiannually, we obtain $(1 + (r/2))^{2t}$. But suppose the compounding process is carried to days, to hours, to seconds, and so on. Suppose the compounding were carried to the ultimate end by compounding continuously. What would \$1.00 amount to after t years? The answer, which we are not sophisticated enough to derive or even intuit, turns out to be e^{tr}.

For an example, if \$10 is invested at a rate of 0.05 for 2 years, continuously compounded, the amount at the end of 2 years is

$$10e^{2(0.05)}$$

Using Table I at the back of the book $e^{2(0.05)} = e^{0.1} = 1.105$. Therefore, \$10 invested at a rate of 0.05 yields $10(1.105) = \$11.05$ under continuous compounding.

: What is the yearly advantage to a depositor of having his money invested at a rate of 0.06 compounded continuously versus compounded semiannually?

Answer. Compounding semiannually gives $(1 + 0.03)^2 = (1.03)(1.03) = 1.0609$. Compounding continuously gives $e^{1(0.06)} = e^{0.06} = 1.0618$. The difference is 0.0009 per dollar or 9 cents per \$100. This calculation suggests that it is advantageous to use continuous compounding to approximate the results of daily or even weekly compounding. The approximation is sufficiently close for all practical purposes. After all, $e^{0.06}$ is much easier to evaluate than the precise answer to daily compounding at a 0.06 annual rate, which is given by $(1 + (0.06/365))^{365}$.

4-2 RULES FOR LOGARITHMS

In Section 2–1 we studied several rules for operating with exponents. Related rules exist for logarithms. These rules can be derived using what we already know about exponents and the inverse functional relationship between logarithms and exponents.

Suppose that we desire the logarithm of (4)(16) to the base 2. That is, we wish to find

$$y = \log_2 [(4)(16)].$$

Using the exponential inverse and the first rule of exponents we write

$$\begin{aligned} 2^y &= (4)(16) = (2^2)(2^4) \\ &= 2^6. \end{aligned}$$

Hence $y = 6$, since the base is the same. But this is the same answer we get by writing

$$y = \log_2 (4) + \log_2 (16) = 2 + 4 = 6,$$

using the definition of a logarithm as the power needed to raise the base, 2 in this case, to the given number, first 4 and then 16 here. As a check, $(4)(16) = 64 = 2^6$. We have illustrated (but not proved) that, corresponding to the first rule for exponents, we can write the following first rule for logarithms.

Rule 1 for Logarithms: PRODUCTS

The logarithm of a product is equal to the sum of the logarithms. In symbolic form:

$$\begin{aligned} y &= \log_b(a_1 \cdot a_2 \cdot \ldots \cdot a_i \cdot \ldots \cdot a_n) \\ &= \log_b a_1 + \ldots + \log_b a_i + \ldots + \log_b a_n \\ &= \sum_{i=1}^{n} \log_b a_i. \end{aligned}$$

Suppose that the logarithmic expression desired is

$$y = \log_2 (2^3).$$

Again, using the inverse relationship, we write

$$2^y = 2^3.$$

Hence $y = 3$. But since $\log_2(2) = 1$, that is, 1 is the power necessary to raise 2 to 2, we can write

$$y = 3(1) = 3 \log_2 2.$$

This illustrates, but again it does not prove, the second rule for logarithms.

Rule 2 for Logarithm: POWERS

The logarithm of any term raised to a power is equal to the power multiplied by the logarithm of the term. In symbolic form:

$$\log_b a^n = n \log_b a.$$

Perhaps another development of this expression is useful. Again, using a numerical example, see if you can follow the algebra. Given

$$8 = 2^3.$$

Now taking logarithms of each side to the base 2,

$$\log_2 8 = \log_2 (2^3)$$

or

$$\log_2 8 = 3$$

since, on the right, 3 is, by definition, the power one must raise the base 2 to get 2^3. Substituting $\log_2 8$ for 3 in the initial equation yields

$$8 = 2^{\log_2 8}.$$

Squaring both sides and then using the second rule for exponents,

$$8^2 = (2^{\log_2 8})^2 = 2^{2 \log_2 8}.$$

Using the definition of a logarithm, $2 \log_2 8$ is the power to which 2 must be raised to give 8^2. Hence $2 \log_2 8$ is the logarithm of 8^2 to the base 2. We write

$$\log_2 8^2 = 2 \log_2 8,$$

which agrees with the second rule for logarithms. The example is complete.

Consider now the logarithm of the quotient of two expressions. For example,

$$y = \log_3(\tfrac{27}{9}) = \log_3 (3) = 1.$$

Using rule 3 for exponents from Section 2–1, we may write

$$y = \log_3 [(27)(9^{-1})],$$

and using rule 1 for logarithms,

$$y = \log_3 (27) + \log_3 (9^{-1}).$$

Now using rule 2 for logarithms,

$$y = \log_3 (27) - \log_3 (9).$$

This is equal to 3 − 2, or 1, as we previously saw, and hence the manipulations have not altered the correct answer. This example suggests rule 3 for logarithms.

Rule 3 for Logarithms: QUOTIENTS

The logarithm of a quotient is equal to the logarithm of the numerator minus the logarithm of the denominator. In symbolic form:

$$\log_b \left(\frac{a}{c}\right) = \log_b a - \log_b c.$$

: See if you can answer the following.

1. $\log_4 (2) =$
2. $\log_r (3^0) =$
3. $\log_2 (8^{-1}) =$
4. $\log_3 (81^{1/4}) =$

Answers

1. The first question asks for the power needed to raise 4 to yield 2; the answer is the $\frac{1}{2}$ power.
2. Using the second rule for exponents, we get $0 \log_r 3$, and zero times anything is zero.
3. Again using rule 2 we get $(-1) \log_2 8 = -3$.
4. Working inside the parentheses, $81^{1/4}$ is the fourth root of 81. This is 3. Hence $\log_3 (81^{1/4}) = \log_3 (3) = 1$.

4-3 GRAPHS OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS

The exponential function given by

$$y = b^x, \qquad b > 1,$$

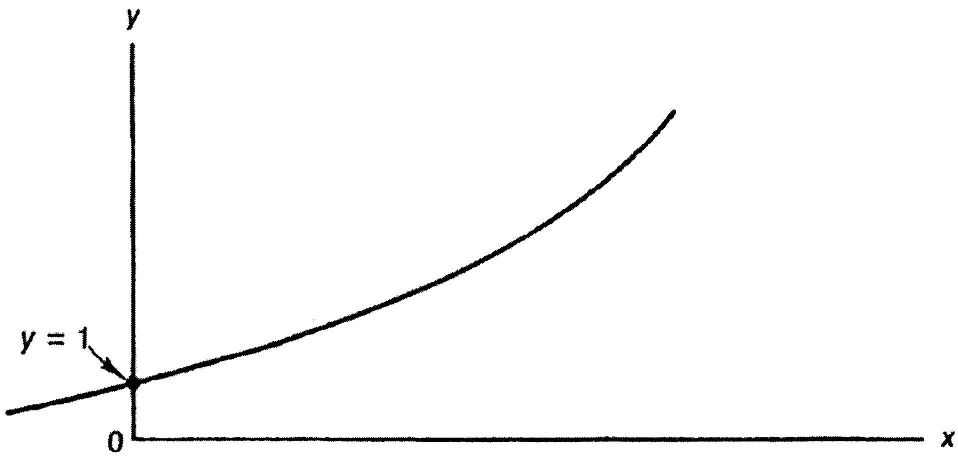
has the logarithmic equivalent, using the concept of inverse functions, given by

$$x = \log_b y.$$

This function is graphed in Figure 4–1 using the normal Cartesian axes. The reader will note that the function can be viewed as a growth curve for non-negative values of x. The growth is a constant percentage. For example, if $b = 2$, a partial sequence (for $x = 0, 1, 2, \ldots$) is $y = 2^x = 1, 2, 4, 8 \ldots$. This series increases by 100 percent; it doubles for each unit increase in x. A frequently encountered member of this family of functions is where $b = e = 2.71828 \ldots$. This gives the function $y = e^x$.

FIGURE 4–1

Graph of $y = b^x$ or $x = \log_b y$



A related but different growth curve is obtained by interchanging the x and y values. This yields

$$x = b^y, \qquad b > 1,$$

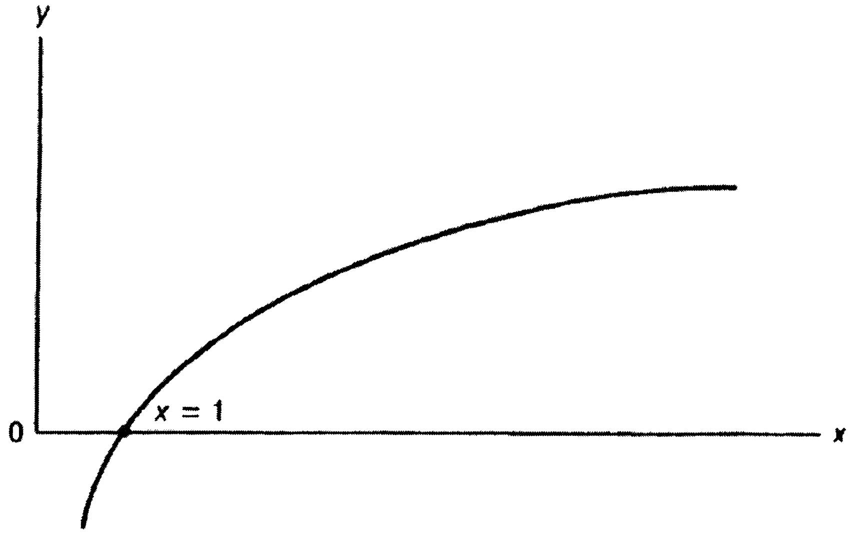
or in inverse form,

$$y = \log_b x.$$

The graph of this function is given in Figure 4–2. This function is usually relevant to managerial problems only for $x \geq 1$. Growth in this case continues but, in contrast to that in Figure 4–1, by decreasing amounts.

FIGURE 4–2

Graph of $x = b^y$ or $y = \log_b x$ for $b > 1$



Reading graphs and obtaining values for situations involving curves of the type just described is often easier if the equations are transformed mathematically to straight lines. In the first example, if we consider

$$y = b^x$$

and take logarithms of each side to an arbitrary base, say r, we obtain

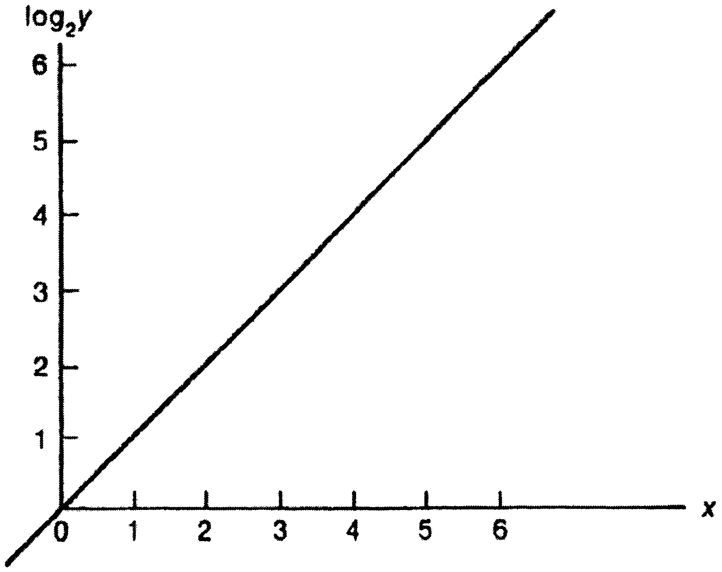
$$\log_r y = x \log_r b.$$

Since $\log_r b$ is a constant, this equation will plot as a straight line through the origin, if we plot the logarithm of y to the base r on the y axis. This is illustrated in Figure 4–3. The illustration assumes for simplicity that $r = b = 2$. Thus the function can be written $y = 2^x$. Now if instead of plotting y, we plot $\log_2 y$, the following points appear on the graph:

x	y	$\log_2 y$
0	1	0
1	2	1
2	4	2
3	8	3
4	16	4
.	.	.

FIGURE 4–3

Graph of $\log_2 y$ where $b = 2$ and $y = b^x$



Mathematicians call the graph used in Figure 4–3 a semilogarithmic graph since the logarithms of one of the variables, y, is plotted rather than the values of the variable itself. In this case the vertical scale was transformed. A similar approach could be used on the horizontal scale to transform the exponential equation $x = b^y$ to a linear graph. Sometimes it is necessary to convert both scales to logarithms to obtain a linear relationship. An example is the equation $y = x^b$.

Logarithmic transformations to achieve linearity are often useful in curve-fitting problems, particularly when the data reflect exponential growth. Fitting a curve to population-growth or sales-growth curves is a common example where taking logarithms of the data may be helpful.

The actual data points may not lie on a straight line even after taking logarithms or using some other appropriate procedure to transform the original data. If we still wish to select a single linear equation to represent the data, then we need a technique that selects the "best" linear representation. One means of making this selection is the method of least-squares. This technique is discussed in statistics courses and we will not attempt a rigorous discussion here. However, the technique is consistent with the following six steps:

1. Select a line $y = a + bx$.
2. For each data point with coordinates x', y', solve the equation $\hat{y} = a + bx'$.
3. Compute the difference $y' - \hat{y}$ for each data point.
4. Square each of these differences to obtain $(y' - \hat{y})^2$.
5. Sum these squared differences to obtain $\Sigma(y' - \hat{y})^2$.
6. Select that line which produces the smallest (i. e., minimizes the) sum.